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A GEOMETRICAL INTERPRETATION OF THE GENERALIZED LAW OF THE MEAN

By Charles N. Haskins

The geometrical interpretation of the ordinary Law of the Mean,

$$f(b) - f(a) = (b - a) f'(X), \qquad a < X < b, \tag{1}$$

and of the function

$$\phi(x) \equiv \frac{f(b) - f(a)}{b - a}(x - a) - [f(x) - f(a)], \tag{2}$$

by which its proof is made to depend on Rolle's theorem, are well-known and may be stated as follows. Consider the curve $y = f(x)$, the points $A: \{a, f(a)\}$, $B: \{b, f(b)\}$ upon it, and the chord AB. Then there exists a point $P: \{X, f(X)\}$ on the curve between A and B such that the tangent at P is parallel to the chord AB. The function $\phi(x)$ represents the distance from the variable point $\{x, f(x)\}$ of the curve to the chord, this distance being measured parallel to the axis of Y.

The importance of the generalized law of the mean, especially in the discussion of indeterminate forms, is well recognized,[*] but its simple geometric interpretation and that of the function by means of which its proof is made to depend on Rolle's theorem have not been emphasized as strongly as is desirable if the law is to be useful in elementary instruction in the calculus.

The Generalized Law of the Mean is

$$\frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(X)}{F'(X)}, \qquad a < X < b. \tag{3}$$

The function $\Phi(x)$ used in the proof is

$$\Phi(x) \equiv \frac{f(b) - f(a)}{F(b) - F(a)}[F(x) - F(a)] - [f(x) - f(a)]. \tag{4}$$

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[*] Cf. Osgood, Annals of Mathematics, ser. 1, vol. 12, p. 65, 1899; and *Differential and Integral Calculus*, p. 280.

The interpretations are as follows. Consider the space curve, $y = f(x)$, $z = F(x)$, the points $A: \{a, f(a), F(a)\}$, $B: \{b, f(b), F(b)\}$ upon it, the chord AB, and the plane determined by the chord AB and the straight line through A parallel to the axis of X. Call this plane the *chordal plane*. Then as interpretation of the law (3) we have:

There exists a point $P: \{X, f(X), F(X)\}$ *on the curve such that the tangent at* P *is parallel to the chordal plane.*

And as interpretation of the function $\Phi(x)$ in (4):

The function $\Phi(x)$ *represents the distance from the variable point* $\{x, f(x), F(x)\}$ *of the curve to the chordal plane, this distance being measured parallel to the axis of* Y.

The proofs of these statements are at once obvious if it is noted that the equation of the chordal plane is

$$y = \frac{f(b) - f(a)}{F(b) - F(a)} [z - F(a)] + f(a).$$

When once this geometric representation of the law and of the auxiliary function $\Phi(x)$ has been obtained, the reason for the appearance of this function is clear, and the course of the proof is seen to be identical with that of the simpler law (1).

The more general forms of the Law of the Mean expressed by*

$$\begin{vmatrix} \phi'(T) & f'(T) & F'(T) \\ \phi(a) & f(a) & F(a) \\ \phi(b) & f(b) & F(b) \end{vmatrix} = 0, \qquad a < T < b;$$

and

$$\begin{vmatrix} \phi'(T_i) & f'(T_i) & F'(T_i) & 0 \\ \phi(a) & f(a) & F(a) & 1 \\ \phi(b) & f(b) & F(b) & 1 \\ \phi(c) & f(c) & F(c) & 1 \end{vmatrix} = 0, \qquad \begin{matrix} i = 1, 3, \\ a < T_1 < b < T_3 < c; \end{matrix}$$

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[*] Cf. Jordan, *Cours d'Analyse*, vol. 1, p. 66.

$$\begin{vmatrix} \phi''(T_2) & f''(T_2) & F''(T_2) & 0 \\ \phi(a) & f(a) & F(a) & 1 \\ \phi(b) & f(b) & F(b) & 1 \\ \phi(c) & f(c) & F(c) & 1 \end{vmatrix} = 0, \qquad a < T_1 < T_2 < T_3 < c;$$

admit a similar interpretation by means of the space curve: $x = \phi(t)$, $y = f(t)$, $z = F(t)$. The first expresses the fact that if

$$A : \{\phi(a), f(a), F(a)\}, \qquad B : \{\phi(b), f(b), F(b)\}$$

are two points of a space curve, there is between A and B a point T on the curve where the tangent is parallel to the plane determined by A, B, and the origin. The second and third state that if A, B, C, are three non-collinear points of a space curve, there is a point T_1 on the curve between A and B, and a point T_3 between B and C, where the tangent to the curve is parallel to the plane $A\ B\ C$; and further there is a point T_2 between T_1 and T_3 where the principal normal is parallel to that plane.

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DECEMBER, 1907.